Math 155, ***Lecture Notes- Bonds*** **Name____________**

Section 9.1 *Sequences*

A **sequence** is a function whose domain is the set of positive integers. It will usually be denoted with subscript notation rather than function notation. You can use your graphing calculator in "sequence mode" to plot terms and create tables that show terms in a sequence.

For example:

$f(1) = a_1$ or $a(1) = a_1$

$f(2) = a_2$ or $a(2) = a_2$

$f(3) = a_3$ or $a(3) = a_3$

$\vdots$

$f(n) = a_n$ or $a(n) = a_n$

a_1 – first term

a_2 – second term

$\vdots$

a_n – n^{th} term

a_{n+1} – $(n+1)^{st}$ term

$\vdots$

An entire sequence can be denoted as $\{a_n\}$.

Ex. 1:

$$\{a_n\} = \left\{1 - \frac{1}{n}\right\} = \left\{0,\ \frac{1}{2},\ \frac{2}{3},\ \frac{3}{4},\ \frac{4}{5},\ \ldots\right\}$$

Ex. 2:

$$\{a_n\} = \left\{(-1)^n n\right\} = \{0,\ -1,\ 2,\ -3,\ 4,\ \ldots\}$$

Some sequences are *recursively defined*.

Ex. 3:

$\{d_n\}$ is defined as $d_{n+1} = d_n - 5$ and $d_1 = 25$.

So, $d_2 = d_1 - 5$ | $d_3 = d_2 - 5$ | $d_4 = d_3 - 5$

$d_2 = 25 - 5$ | $d_3 = 20 - 5$ | $d_4 = 15 - 5$

$d_2 = 20$ | $d_3 = 15$ | $d_4 = 10$

etc . . . .

For the majority of the chapter, we'll be looking at sequences that have limiting values. These sequences are said to **converge**.

Ex. 4:

$$\{a_n\} = \left\{\frac{1}{2^n}\right\} = \left\{\frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \frac{1}{16},\ \frac{1}{32},\ldots\right\}$$

This sequence converges to 0.

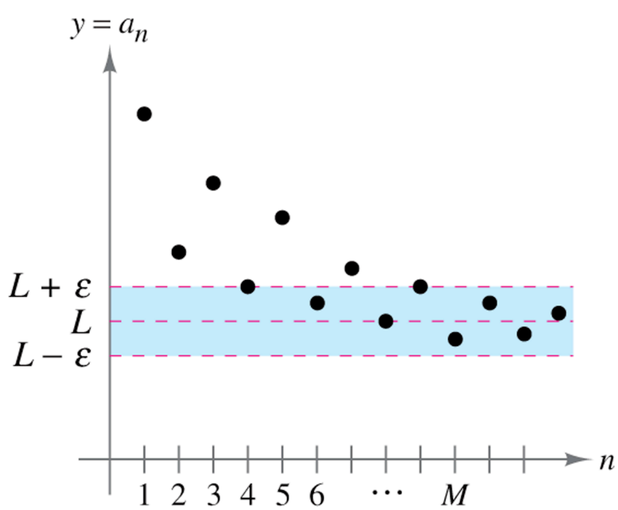
Definition of the Limit of a Sequence

Let L be a real number. The **limit** of a sequence $\{a_n\}$ is L, written as

$$\lim_{n\to\infty} a_n = L$$

if for each $\varepsilon > 0$, there exists $M > 0$ such that $|a_n - L| < \varepsilon$ whenever $n > M$. If the limit L of a sequence exists, then the sequence **converges** to L. If the limit of a sequence does not exist, then the sequence **diverges.**

If we plot the terms of a convergent sequence, we will see a "horizontal asymptote." That is, we will see the sequence exhibit asymptotic behavior.



Ex. 5:

Given: $\{a_n\} = \left\{\dfrac{n+4}{n+1}\right\}$

Consider $\lim\limits_{n\to\infty} a_n =$

This sequence converges to 1.

THEOREM 9.1 Limit of a Sequence

Let L be a real number. Let f be a function of a real variable such that

$$\lim_{x \to \infty} f(x) = L.$$

If $\{a_n\}$ is a sequence such that $f(n) = a_n$ for every positive integer n, then

$$\lim_{n \to \infty} a_n = L.$$

In other words, if a sequnce $\{a_n\}$ "agrees" with a function f at every positive integer, and if $f(x) \to L$ as $x \to \infty$, then $\{a_n\} \to L$ as well.

Ex. 6:

Given: $\{a_n\} = \left\{\left(1 + \frac{1}{n}\right)^n\right\}$ Consider $\lim\limits_{n \to \infty} a_n =$

THEOREM 9.2 Properties of Limits of Sequences

Let $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = K$.

1. $\lim_{n\to\infty} (a_n \pm b_n) = L \pm K$
2. $\lim_{n\to\infty} ca_n = cL$, c is any real number
3. $\lim_{n\to\infty} (a_n b_n) = LK$
4. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{K}$, $b_n \neq 0$ and $K \neq 0$

New Notation: Factorial !

Try working with these on your graphing calculator.

$n! = 1\cdot 2\cdot 3\cdot 4 \cdots (n-1)\cdot n$

$0! = 1$

$3! = 1\cdot 2\cdot 3 = 6$

$5! = 1\cdot 2\cdot 3\cdot 4\cdot 5 = 120$

$2n! = 2(n!) = 2\cdot[1\cdot 2\cdot 3\cdot \cdots (n-1)\cdot n]$

$(2n)! = 1\cdot 2\cdot 3\cdot 4 \cdots (n-1)\cdot n\,(n+1) \cdots (2n-1)\cdot 2n$

$4! = 1\cdot 2\cdot 3\cdot 4 = 24$

$6! = 1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6 = 720$

$2! = 1\cdot 2 = 2$

$1! = 1 = 1$

THEOREM 9.3 Squeeze Theorem for Sequences

If

$$\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} b_n$$

and there exists an integer N such that $a_n \le c_n \le b_n$ for all $n > N$, then

$$\lim_{n\to\infty} c_n = L.$$

Ex. 7:

Given: $\{a_n\} = \left\{\dfrac{\sin(n)}{n}\right\}$ Consider $\lim\limits_{n\to\infty} a_n =$

THEOREM 9.4 Absolute Value Theorem

For the sequence $\{a_n\}$, if

$$\lim_{n\to\infty} |a_n| = 0 \qquad \text{then} \qquad \lim_{n\to\infty} a_n = 0.$$

Ex. 8:

Given: $\{a_n\} = \left\{\dfrac{5n}{\sqrt{n^2+4}}\right\}$ Consider $\lim\limits_{n\to\infty} a_n =$

Ex. 9:

Given: $\{a_n\} = \left\{\frac{(n-2)!}{n!}\right\}$ Consider $\lim_{n\to\infty} a_n =$

Ex. 10:

Given: $\{a_n\} = \left\{\dfrac{n^2}{2n+1} - \dfrac{n^2}{2n-1}\right\}$ Consider $\lim\limits_{n\to\infty} a_n =$

Ex. 11:

Given: $\left\{a_n\right\} = \left\{\cos(\pi n)\right\}$ Consider $\lim\limits_{n\to\infty} a_n =$

Ex. 12:

Given: $\{a_n\} = \left\{\dfrac{1\cdot 3\cdot 5\cdot 7\cdots(2n-1)}{n!}\right\}$ Consider $\lim\limits_{n\to\infty} a_n =$

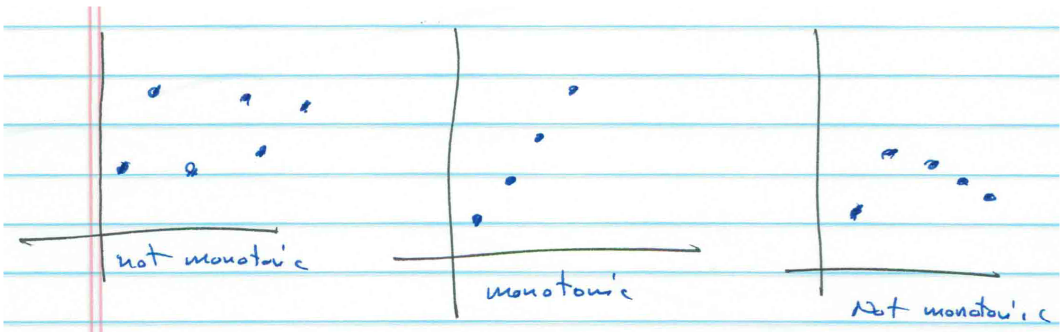
Definition of a Monotonic Sequence

A sequence $\{a_n\}$ is **monotonic** if its terms are nondecreasing

$$a_1 \leq a_2 \leq a_3 \leq \cdot \cdot \cdot \leq a_n \leq \cdot \cdot \cdot$$

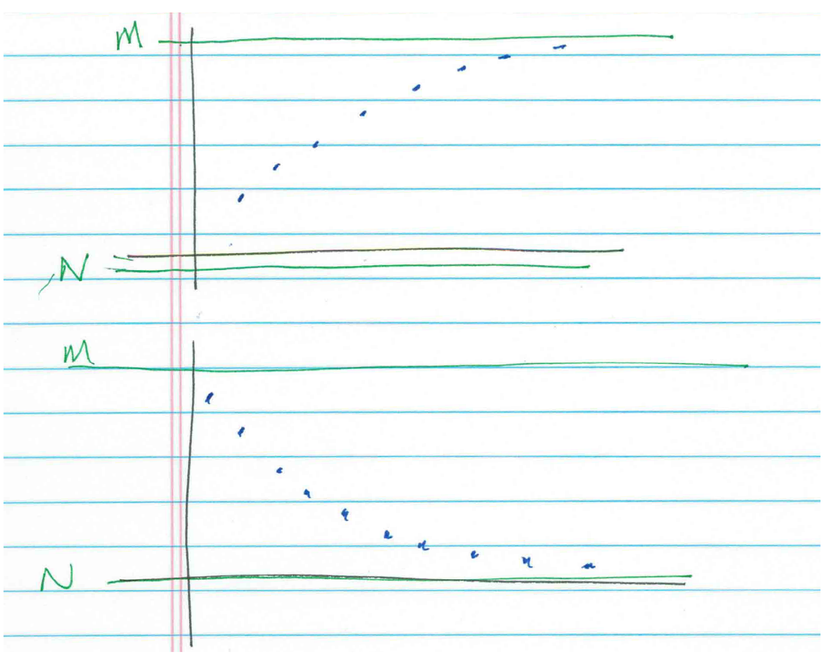
or if its terms are nonincreasing

$$a_1 \geq a_2 \geq a_3 \geq \cdot \cdot \cdot \geq a_n \geq \cdot \cdot \cdot .$$



Definition of a Bounded Sequence

1. A sequence $\{a_n\}$ is **bounded above** if there is a real number M such that $a_n \leq M$ for all n. The number M is called an **upper bound** of the sequence.
2. A sequence $\{a_n\}$ is **bounded below** if there is a real number N such that $N \leq a_n$ for all n. The number N is called a **lower bound** of the sequence.
3. A sequence $\{a_n\}$ is **bounded** if it is bounded above and bounded below.

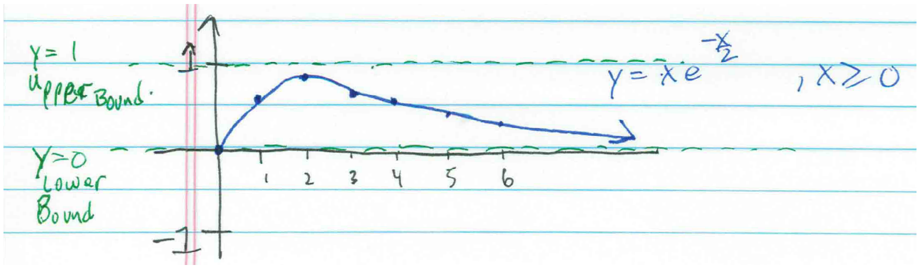


THEOREM 9.5 Bounded Monotonic Sequences

If a sequence $\{a_n\}$ is bounded and monotonic, then it converges.

Ex. 13:

Given: $\{a_n\} = \left\{ n e^{\frac{-n}{2}} \right\}$ Consider $\lim\limits_{n\to\infty} a_n =$



From the graph of $y = xe^{\frac{-x}{2}}$, for $x \geq 0$, we can see that the function is bounded above by $y = 1$ and bounded below by $y = 0$. Therefore, by Theorem 9.5, $\left\{ne^{\frac{-n}{2}}\right\}$ is a convergent sequence, since $\left\{ne^{\frac{-n}{2}}\right\}$ is bounded and monotonic for $n \geq 2$.

Ex. 14: *The Fibonacci Sequence*
Consider the sequence is defined by $a_{n+2} = a_{n+1} + a_n$ with $a_1 = 1$ and $a_2 = 1$.

$$\{a_n\} = \{1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ .....\}$$

This is the Fibonacci Sequence.